NUMERICAL CALCULATION OF THE POTENTIAL OF A RING OF CHARGE

I. Introduction

The general formula for the potential at a point P located at (x, y, z) due to a line charge distribution is

$$V(P) = \frac{1}{4\pi\varepsilon_o} \int_C \frac{\rho_\ell}{|\vec{R}-\vec{R}'|} d\ell' \tag{1}$$

where $\vec{R} = x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector to the observation point and $\vec{R}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$ is a position vector to the source point. As illustrated by the examples in Chang, the potential due to a few special charge distributions can be reduced to closed form expressions, but in most cases the integration must be done numerically. The objective of this lab is to obtain the potential of a ring of charge at an arbitrary location in space by performing the integration using the rectangular rule.

II. Potential of a Ring of Charge

Consider the ring charge shown in Figure 1. Since the ring conforms to the cylindrical variable $r = \text{constant}$, calculations are most easily performed in cylindrical coordinates. For a ring with radius b

$$V(P) = \frac{1}{4\pi\varepsilon_o} \int_0^{2\pi} \frac{\rho_\ell}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} b\, d\phi' \tag{2}$$

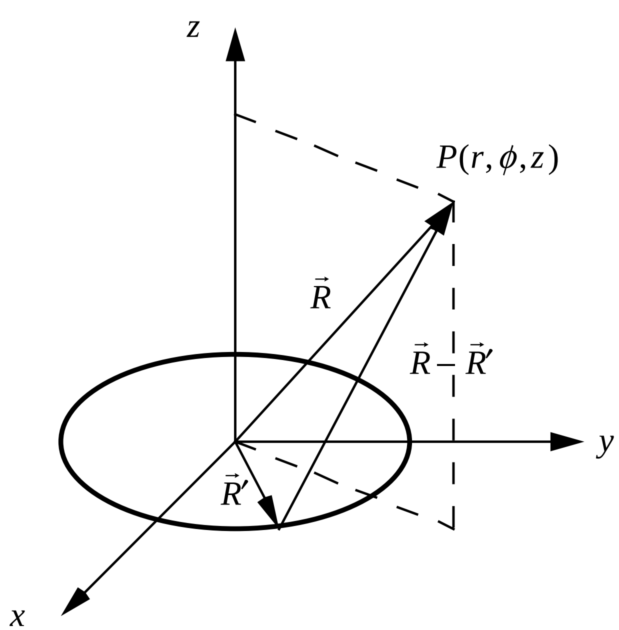


Figure 1: Ring charge.

In cylindrical coordinates

$$x' = b\cos\phi' \qquad (3)$$
$$y' = b\sin\phi'$$

Thus equation (1) can be expressed as

$$V(P) = \frac{b}{4\pi\varepsilon_o}\int_0^{2\pi} f(\phi')d\phi' \qquad (4)$$

where

$$f(\phi') = \frac{\rho_\ell(\phi')}{\sqrt{(x - b\cos\phi')^2 + (y - b\sin\phi')^2 + z^2}} \qquad (5)$$

III. The Rectangular Rule

The integral with respect to ϕ' can be evaluated using the rectangular rule. The integral is approximated by the sum of the areas under a step approximation of the actual integrand as illustrated in Figure 2. The integration interval from 0 to 2π is divided into N subintervals of equal length

$$\Delta\phi = 2\pi / N \qquad (6)$$

The midpoints of the intervals are located at

$$\phi_n = (2n-1)\frac{\Delta\phi}{2},\ n = 1,\ldots,N \qquad (7)$$

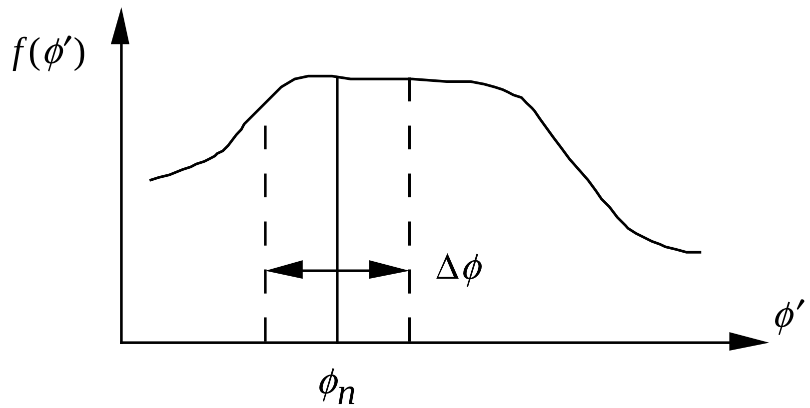


Figure 2: Integration interval and midpoint.

The level of the step can be determined in several ways. One is to use an average of the values at the ends of the interval, $\phi_n - \frac{\Delta\phi}{2}$ and $\phi_n + \frac{\Delta\phi}{2}$. However, in this case we will simply use the value of $f(\phi')$ at the midpoint of each interval

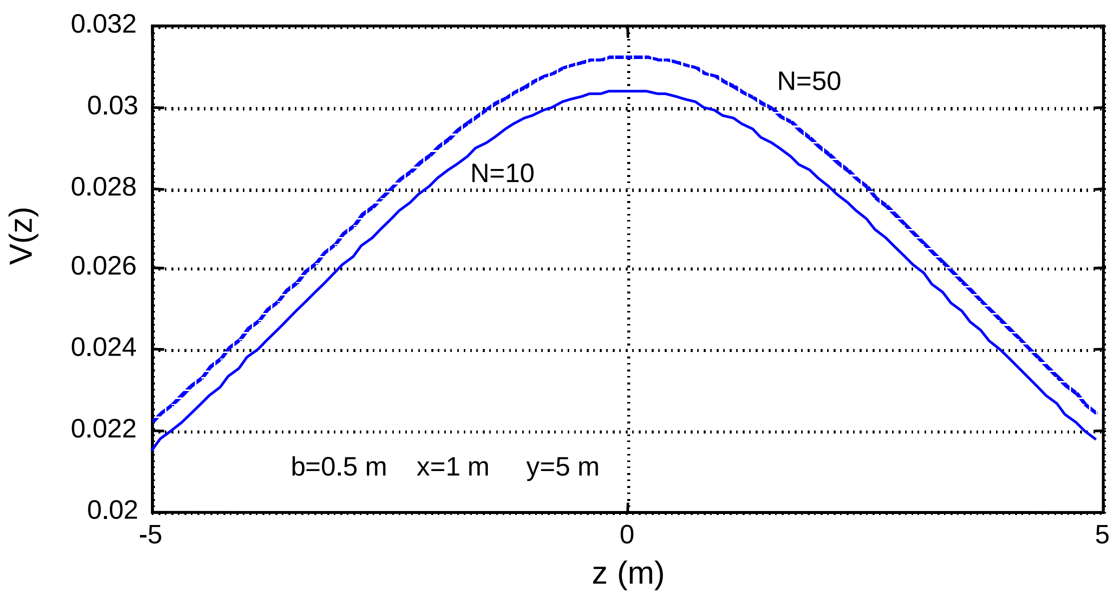
$$\int_{\phi_n-\frac{\Delta\phi}{2}}^{\phi_n+\frac{\Delta\phi}{2}} f(\phi')d\phi' \approx \Delta\phi\, f(\phi_n) \tag{8}$$

so that equation (2) becomes

$$V(x,y,z) = \frac{b\Delta\phi}{4\pi\varepsilon_o}\sum_{n=1}^{N}\frac{\rho_\ell(\phi_n)}{\sqrt{(x-b\cos\phi_n)^2+(y-b\sin\phi_n)^2+z^2}} \tag{9}$$

Finally, if the charge density is constant over the entire ring then

$$V(x,y,z) = \frac{\rho_\ell b\Delta\phi}{4\pi\varepsilon_o}\sum_{n=1}^{N}\frac{1}{\sqrt{(x-b\cos\phi_n)^2+(y-b\sin\phi_n)^2+z^2}} \tag{10}$$



Sample result for $\rho_\ell(\phi') = |\sin(4\phi')|$.

IV. Exercises

Computer code equation (9) in Matlab, Mathcad, Mathematica or other high-level programming language.

(1) Plot the potential as a function of position along the z axis for the following parameters:

$x = y = 0, -5\,\mathrm{m} \le z \le 5\,\mathrm{m},\ b = 0.5\,\mathrm{m},\ \rho_\ell = 1\,\mathrm{C/m}$

(2) Verify your result by comparing the computed data with the closed form result obtained in class

$$V(0,0,z) = \frac{b\rho_\ell}{2\varepsilon_o\sqrt{b^2+z^2}}$$

(3) Plot the potential of a sinusoidal ring charge of the form

$$\rho_\ell(\phi') = \left|\sin(5\phi')\right|$$

as a function of z for $x = 1\,\mathrm{m},\ y = 5\,\mathrm{m}$. Check the convergence of the solution by comparing the results for 5, 20 and 50 integration intervals.

(4) Include a copy of your program in your lab report.